

Signals in Natural Domain

Chapter 4 : Time-Domain Representation for Linear Time-Invariant Systems

In this chapter we will consider several methods for describing relationship between the input and output of linear time-invariant systems.

The Convolution Sum:

The representation of discrete time signals in terms of impulses.

The key idea is to express an arbitrary discrete time signal as weighted sum of time shifted impulses.

Consider the product of signal  $\{x[n]\}$  and the impulse sequence. We know that

$$a\{\delta[n]\} = \begin{cases} a, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

and

$$b\{\delta[n-k]\} = \begin{cases} b, & n = k \\ 0, & n \neq k \end{cases}$$

Using these relations we can write

$$\begin{aligned} \{x[n]\} &= \dots x[-2]\{\delta[n+2]\} + x[-1]\{\delta[n+1]\} + x[0]\{\delta[n]\} + x[1]\{\delta[n-1]\} + x[2]\{\delta[n-2]\} + \dots \\ &= \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\} \end{aligned}$$

(4.1)

A graphical illustration is shown below

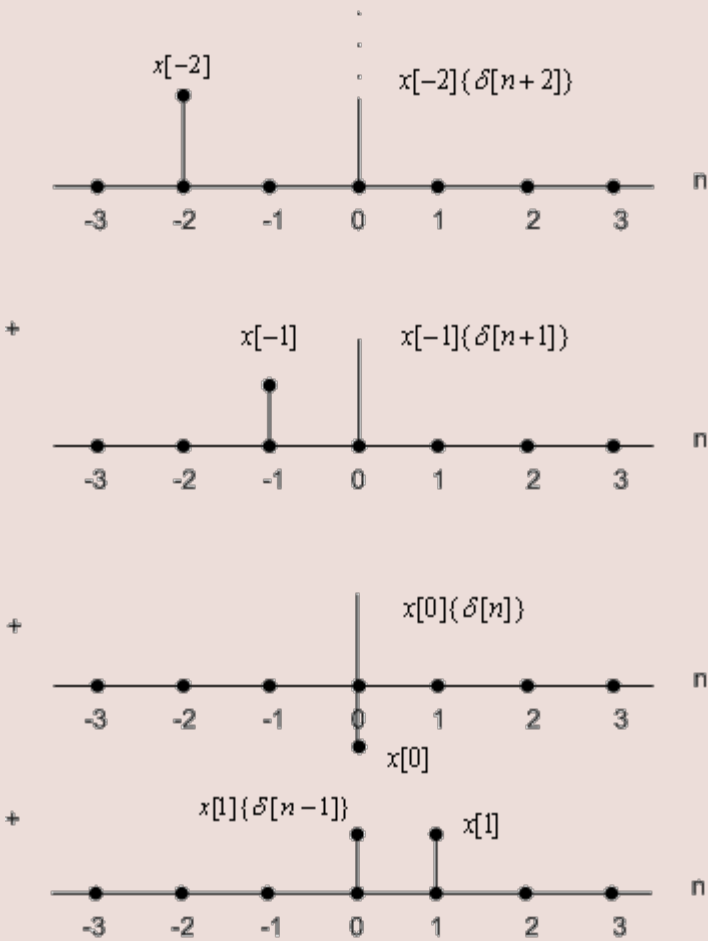


Fig 4.1

Given an arbitrary sequence we can write it as a linear combination of shifted unit impulses  $\{\delta[n-k]\}$ , where the weights of their combination are  $x[k]$ , the  $k^{\text{th}}$  term of the sequence. For any given  $n$ , in the summation

$$\sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}$$

there is only one term which is non-zero and so we do not have to worry about the convergence.

Consider the unit step sequence  $\{u[n]\}$ . Since  $u[n] = 0, n < 0$ , and  $u[n] = 1, n \geq 0$ , it has representation

$$\{u[n]\} = \sum_{k=0}^{\infty} \{\delta[n-k]\}$$



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The Discrete Time Impulse response of linear Time Invariant System:

We use linearity property of the system to represent its response in terms of its response shifted impulse sequences. The time invariance further simplifies their representation. Let  $\{x[n]\}$  be the input signal and  $\{y[n]\}$  be the output sequence, and  $T(\cdot)$  represent the linear system

$$\begin{aligned} \{y[n]\} &= T(\{x[n]\}) \\ &= T\left(\sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}\right) \end{aligned}$$

using (4.1)

Now we use the linearity property of the system we get

$$\begin{aligned} \{y[n]\} &= \sum_{k=-\infty}^{\infty} x[k]T(\{\delta[n-k]\}) \\ \{y[n]\} &= T(\{x[n]\}) \end{aligned}$$

Note that without countable additivity property the last step is not justified (From finite additivity we can not get countable additivity). Let us define

$$\{h_k[n]\} = T(\{\delta[n-k]\})$$

i.e.  $\{h_k[n]\}$  is the response of the system to a delayed unit sample sequence. Then we see

$$\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{h_k[n]\}$$

The output signal is linear combination of the signals.

In general the responses  $\{h_k[n]\}$  need not be related to each other for different values of  $k$ . However, if linear system is also time-invariant, then these responses are related. Let us define impulse response (unit sample response)

$$\{h[n]\} = T(\{\delta[n]\})$$

Then

$$\begin{aligned} \{h_k[n]\} &= T(\{\delta[n-k]\}) \\ &= \{h[n-k]\} \end{aligned}$$

For the LTI system output  $\{y[n]\}$  is given by

$$\{y[n]\} = \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\} \quad (4.2)$$

This result is known as convolution of sequences  $\{x[n]\}$  and  $\{h[n]\}$ . Thus output signal for an LTI system is convolution of input signal  $\{x[n]\}$  and the impulse response. This operation is symbolically represented by

$$\{y[n]\} = \{x[n]\} * \{h[n]\} \quad (4.3)$$

We see that equation (4.2) expresses the response of an LTI system to an arbitrary signal in terms of the system's response to unit impulse. Thus an LTI system is completely specified by its impulse response.

The  $n^{\text{th}}$  term  $y[n]$  in the equation (4.2) is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad (4.4)$$

This is known as convolution sum. To convolve two sequences, we have to calculate this convolution sum

for all values of  $n$ . Since right hand side is sum of infinite series, we assume that this sum is well defined.

Example:

Consider  $\{x[n]\}$  and  $\{h[n]\}$  shown below

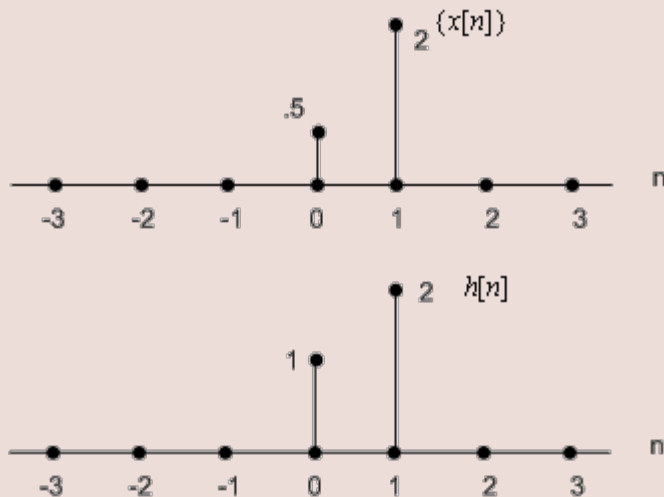


Fig 4.2

Since only  $x[0]$  and  $x[1]$  are non zero we have

$$\{y[n]\} = x[0]\{h[n]\} + x[1]\{h[n-1]\}$$

These are illustrated below

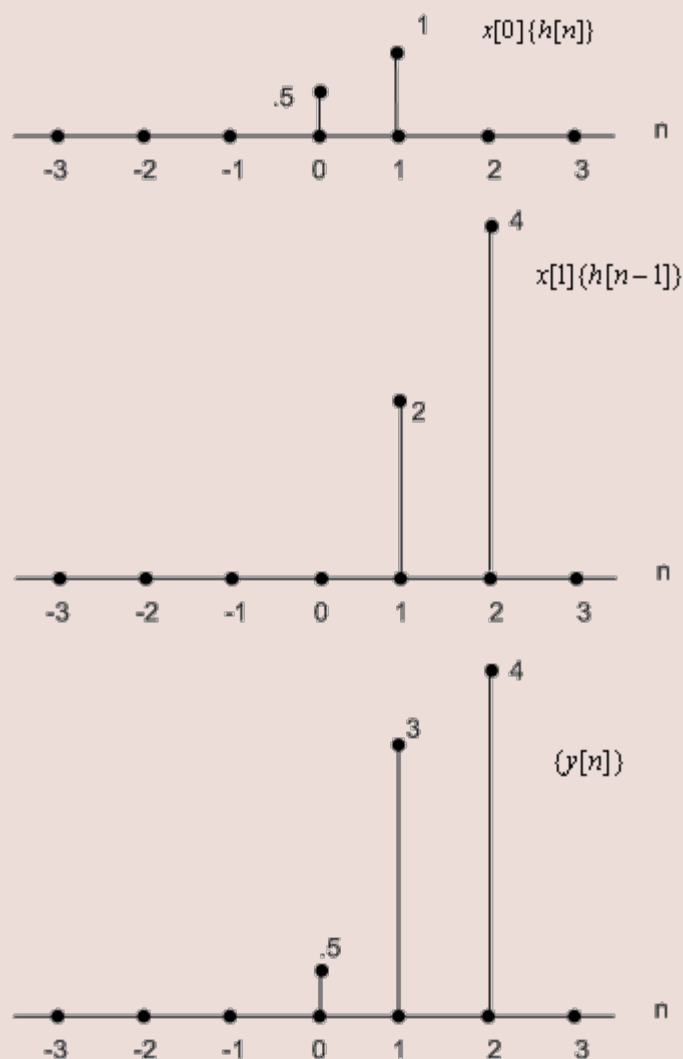


Fig 4.3

Here we have done calculation according to equation (4.2).  
To do calculation according to equation (4.4) we first plot -  $\{x[k]\}$  as function of  $k$  and  $\{h[n-k]\}$  as function of  $k$  for some fixed values of  $n$ . Then multiply sequence  $\{x[k]\}$  and  $\{h[n-k]\}$  term by term to obtain sequence. Then find the sum of the terms of the sequence. This is illustrated below

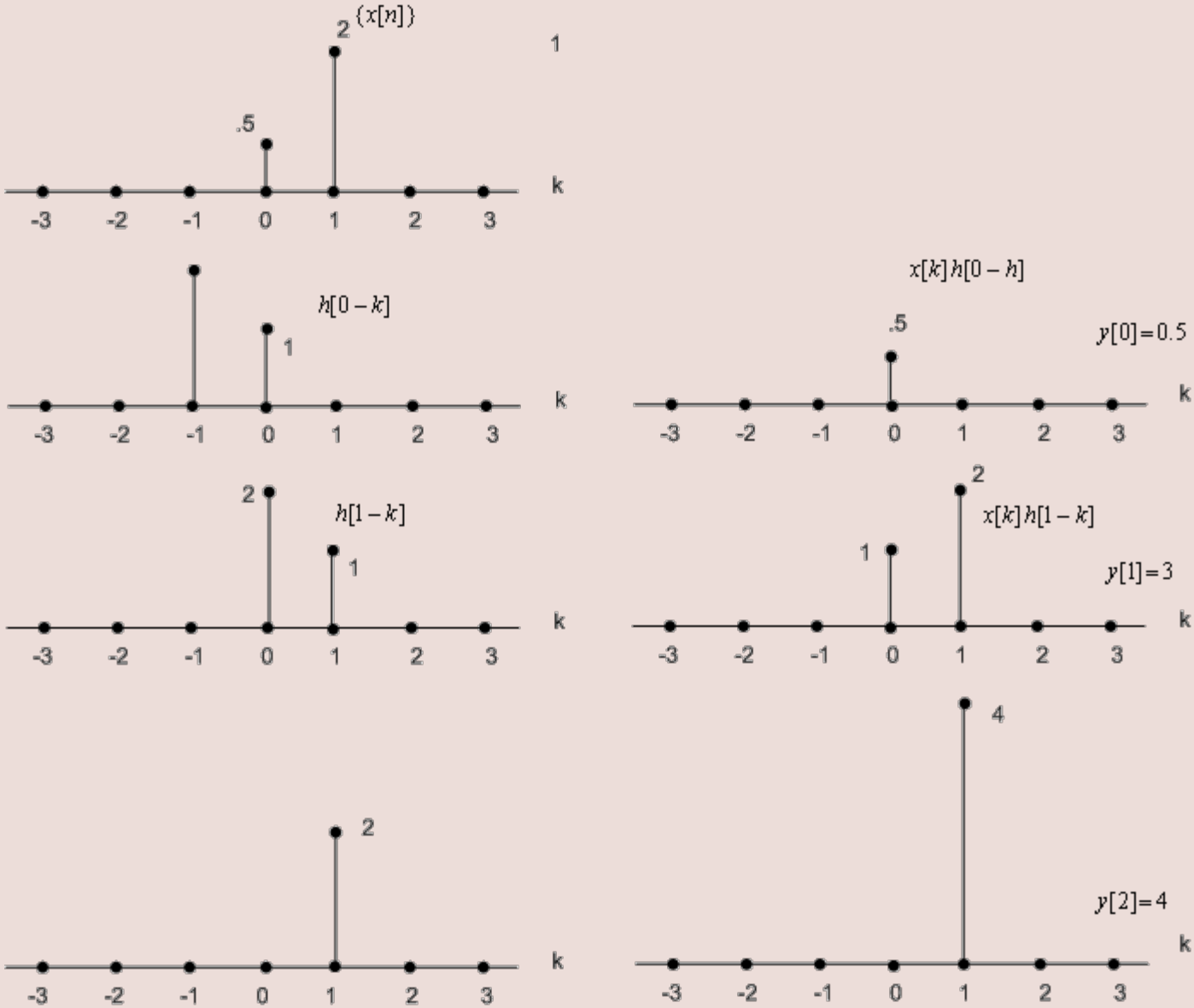


Fig 4.4

One can see easily that for other value of  $n$   $\{x[h]h[h-k]\}$  is all zero sequence and for these value of  $n$ , output is zero.

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Properties of discrete-time linear convolution and system properties

If  $\{x[n]\}$ ,  $\{y[n]\}$  and  $\{z[n]\}$  are sequences, then the following useful properties of the discrete time convolution can be shown to be true

1. Commutativity

$$\{x[n]\} * \{y[n]\} = \{y[n]\} * \{x[n]\}$$

2. Associativity

$$\{x[n]\} * (\{y[n]\} * \{z[n]\}) = (\{x[n]\} * \{y[n]\}) * \{z[n]\} \text{ ,}$$

3. Distributivity over sequence addition

$$\{x[n]\} * (\{y[n]\} + \{z[n]\}) = \{x[n]\} * \{y[n]\} + \{x[n]\} * \{z[n]\}$$

4. The identity sequence  $\{\delta[n]\}$

$$\{x[n]\} * \{\delta[n]\} = \{x[n]\}$$

5. Delay operation

$$\{x[n]\} * \{\delta[n - k]\} = \{x[n - k]\}$$

6. Multiplication by a constant

$$\{a x[n]\} * \{y[n]\} = a(\{x[n]\} * \{y[n]\})$$

Note that these properties are true only if the convolution sum (4.4) exists for every  $n$ .

If the input output relation is defined by convolution i.e. if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

For a given sequence  $\{h[n]\}$ , then the system is linear and time invariant. This can be verified using the properties of the convolution listed above. The impulse response of the systems is obviously.

In terms of LTI system, commutative property implies that we can interchange input and impulse response.



Fig 4.5

The distributive property implies that parallel interconnection of two LTI system is an LTI system with impulse response as sum of two impulse responses.

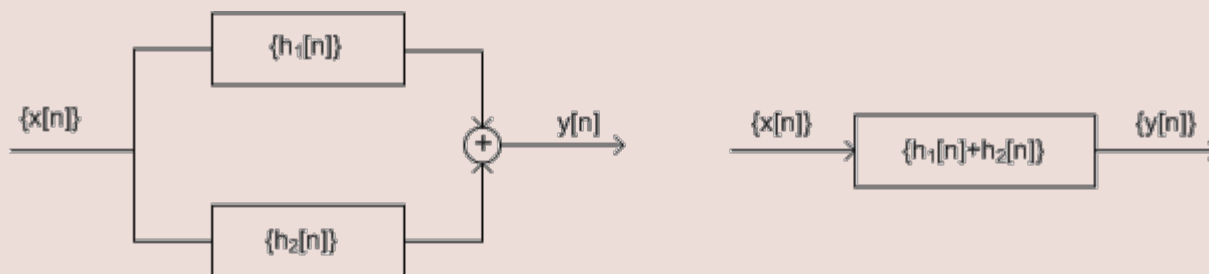


Fig 4.6

The associativity property implies that series connection of two LTI system is an LTI system. Where impulse response is convolution of individual responses. The commutativity property implies that we can interchange the order of the two system in series.

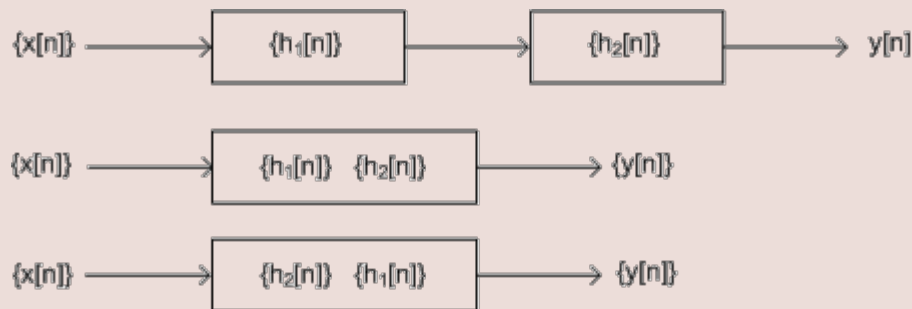


Fig 4.7

Since an LTI system is completely characterized by its impulse response, we can specify system-properties in terms of impulse response.

1. Memoryless system: From equation (4.4) we see that an LTI system is memory less if and only if.
2. Causality for LTI system: The output of a causal system depends only on preset and past-values of the input. In order for a system to be causal  $y[n]$  must not depend on  $x[n]$  for. From equation (4.4) we see that for this to be true, all of the terms  $h[n-k]$  that multiply values of  $x[k]$  for  $k > n$  must be zero.

$$h[n-k] = 0 \text{ for } k > n$$

put  $n-k = m$  to get

$$h[m] = 0 \text{ for } 0 > n-k = m$$

$$\text{or } h[m] = 0 \text{ for } m < 0$$

Thus impulse response  $\{h[n]\}$  for a causal LTI system must satisfy the condition  $h[n] = 0$  for  $n < 0$ .

If the impulse response satisfies this condition, the system is causal. For a causal system we can

$$\text{write } y[n] = \sum_{k=-\infty}^n x[k] h[n-k]$$

$$\text{or } y[n] = \sum_{k=0}^n h[k] x[n-k]$$

We say a sequence  $\{x[n]\}$  is causal if  $x[n] = 0$ , for  $n < 0$ .

3. Stability for LTI system: A system is stable if every bounded input produces a bounded output.

Consider input  $\{x[n]\}$  such that  $|x[n]| < B$  for all  $n$ .

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Taking absolute value

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right|$$

From triangle inequality for complex numbers  $|z_1 + z_2| \leq |z_1| + |z_2|$  we get

$$\begin{aligned}
 |y[n]| &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\
 &= \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|
 \end{aligned}$$

Using property that  $|z_1 z_2| = |z_1| |z_2|$

Since each  $|x[n-k]| < B$  we get

$$\begin{aligned}
 |y[n]| &< \sum_{k=-\infty}^{\infty} |h[k]| B \\
 &= B \sum_{k=-\infty}^{\infty} |h[k]|
 \end{aligned}$$

If the impulse response is absolutely summable, that is

$$\sum_{k=-\infty}^{\infty} |h[k]| = M < \infty \quad (4.5)$$

then  $|y[n]| < MB$

and  $y[n]$  is bounded for all  $n$ , and hence system is stable. Therefore equation (4.5) is sufficient condition for system to be stable. This condition is also necessary. This is prove by showing that if condition (4.5) is violated then we can find a bounded input which produces an unbounded output. Let

$$\sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

Let 
$$x[n] = \begin{cases} \frac{h^*[-n]}{|h[-n]|} & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n] = 0 \end{cases}$$

$$|x[n]| = \begin{cases} 1 & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n] = 0 \end{cases}$$

This is a bounded sequence

$$\begin{aligned}
 y[0] &= \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{\substack{k=-\infty \\ h[k] \neq 0}}^{\infty} h[k]x[0k] \\
 &= \sum_{\substack{k=-\infty \\ h[k] \neq 0}}^{\infty} \frac{h[k]h^*[k]}{|h[k]|} \\
 &= \sum_{k=-\infty}^{\infty} |h[k]| = \infty
 \end{aligned}$$

So  $y[0]$  is unbounded. Thus, the stability of a discrete time linear time invariant system is equivalent to absolute summability of the impulse response.

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#### Causal LTI systems described by difference equations

An important subclass of linear time invariant system is one where the input and output sequences satisfy constant coefficient linear difference equation

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k] \quad (4.6)$$

$$a_k, k = 0, 1, 2, \dots, N$$

$$b_k, k = 0, 1, 2, \dots, M$$

The constants,  $\{x[n]\}$  is input sequence and  $\{y[n]\}$  is output sequence. We can solve equation (4.6) in a manner analogous to the differential equation solution, but for discrete time we can use a different approach. Assume that. We can write

$$y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k] \right\} \quad (4.7)$$

In order to find  $y[n]$  we need previous  $N$  values of the output. Thus if we know the input sequence  $\{x[n]\}$  and a set of initial condition  $y[-N], y[-N+1], \dots, y[-1]$  we can find values of.

Example: Consider the difference equation

$$y[n] + 0.5y[n-1] = x[n]$$

then  $y[n] = x[n] - 0.5y[n-1]$

Let us take  $y[-1] = C$

$$y[0] = x[0] - 0.5C$$

$$y[1] = x[1] - 0.5y[0]$$

$$= x[1] - 0.5x[0] + (0.5)^2 C$$

$$y[2] = x[2] - 0.5y[1]$$

$$= x[2] - 0.5x[1] + (0.5)^2 x[0] + (0.5)^3 C$$

$$y[n] = x[n] - 0.5x[n-1] + (0.5)^2 x[n-2] + \dots + (-0.5)^2 x[0] + (-0.5)^{n+1} C$$

This system is not linear for all values of the initial condition. For a linear system all zero input sequence must produce a all zero output sequence. But if  $C$  is different from zero, then output sequence is not an all system is linear. System is not time invariant in general. Suppose input is  $\{\delta[n]\}$  then we have

$$y[0] = 1 - 0.5C, y[1] = -0.5 + (0.5)^2 C, \dots$$

If we use input as  $\{\delta[n-1]\}$  then

$$y[0] = -0.5C, y[1] = 1 + (0.5)^2 C, \dots$$

It is obvious that second sequence is not a shifted version of the first sequence unless. The system is linear time invariant if we assume initial rest condition, i.e. if  $x[n] = 0, n < n_0$  then. With initial rest condition the system described by constant coefficient-linear difference equation is linear, time invariant and causal. The equation of the form (4.7) is called recursive equation if  $N \geq 1$ , since it specifies a recursive algorithm for finding out the output sequence. In special case  $N = 0$ , we have

$$y[n] = \sum_{k=0}^M \frac{b_k}{a_0} x[n-k] \quad (4.8)$$

Here  $y[n]$  is completely specified in terms of the input. Thus this equation is called non-recursive equation. If input  $\{x[n]\} = \{\delta[n]\}$ , then we see that the output is equal to impulse response

$$y[n] = h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

The impulse response is non-zero for finitely many values. A system with the property that impulse response is non-zero only for finitely many values is known as finite impulse response (FIR) system. A system described by non-recursive equation is always FIR. A system described recursive equation generally has a response which is non-zero for infinite duration and such systems are known as infinite impulse response system (IIR). A system described by recursive equation may have a finite impulse response.

Systems described by constant coefficient linear difference equation can be implemented very easily as we shall see in a later chapter.